Chapter 3
Coherence Properties of the Electromagnetic Field

Abstract In this chapter correlation functions for the electromagnetic field are introduced from which a definition of optical coherence may be formulated. It is shown that the coherent states possess nth-order optical coherence. Photon-correlation measurements and the phenomena of photon bunching and antibunching are described. Phase-dependent correlation functions which are accessible via homodyne measurements are introduced. The theory of photon counting measurements is given.

3.1 Field-Correlation Functions

We shall now consider the detection of an electromagnetic field. A large-scale macroscopic device is complicated, hence, we shall study a simple device, an ideal photon counter. The most common devices in practice involve a transition where a photon is absorbed. This has important consequences since this type of counter is insensitive to spontaneous emission. A complete theory of detection of light requires a knowledge of the interaction of light with atoms. We shall postpone this until a study of the interaction of light with atoms is made in Chap. 10. At this stage we shall assume we have an ideal detector working on an absorption mechanism which is sensitive to the field $E^{(+)}(\boldsymbol{r},t)$ at the space-time point $(\boldsymbol{r},t)$. We follow the treatment of *Glauber* [1].

The transition probability of the detector for absorbing a photon at position $\boldsymbol{r}$ and time t is proportional to

$$T_{if} = |\langle f|E^{(+)}(\boldsymbol{r},\, t)|i\rangle|^2 \tag{3.1}$$

if $|i\rangle$ and $|f\rangle$ are the initial and final states of the field. We do not, in fact, measure the final state of the field but only the total counting rate. To obtain the total count rate we must sum over all states of the field which may be reached from the initial state by an absorption process. We can extend the sum over a complete set of final states since the states which cannot be reached (e.g., states $|f\rangle$ which differ from $|i\rangle$

by two or more photons) will not contribute to the result since they are orthogonal to $E^{(+)}\ (\boldsymbol{r},t)|i\rangle$. The total counting rate or average field intensity is

$$I(\boldsymbol{r},\ t) = \sum_f T_{fi} = \sum_f \langle i|E^{(-)}(\boldsymbol{r},\ t)|f\rangle\langle f|E^{(+)(\boldsymbol{r},\ t)|i\rangle}$$
$$= \langle i|E^{(-)}(\boldsymbol{r},\ t)E^{(+)}(\boldsymbol{r},\ t)|i\rangle\ , \tag{3.2}$$

where we have used the completeness relation

$$\sum_f |f\rangle\langle f| = 1\ . \tag{3.3}$$

The above result assumes that the field is in a pure state $|i\rangle$. The result may be easily generalized to a statistical mixture state by averaging over initial states with the probability P_i, i.e.,

$$I(\boldsymbol{r},\ t) = \sum_i P_i\langle i|E^{(-)}(\boldsymbol{r},\ t)E^{(+)}(\boldsymbol{r},\ t)|i\rangle\ . \tag{3.4}$$

This may be written as

$$I(\boldsymbol{r},\ t) = \mathrm{Tr}\ \{\rho E^{(-)}(\boldsymbol{r},\ t)E^{(+)}(\boldsymbol{r},\ t)\}\ , \tag{3.5}$$

where ρ is the density operator defined by

$$\rho = \sum_i P_i|i\rangle\langle i|\ . \tag{3.6}$$

If the field is initially in the vacuum state

$$\rho = |0\rangle\langle 0|\ , \tag{3.7}$$

then the intensity is

$$I(\boldsymbol{r},\ t) = \langle 0|E^{(-)}E^{(+)}|0\rangle = 0\ . \tag{3.8}$$

The normal ordering of the operators (that is, all annihilation operators are to the right of all creation operators) yields zero intensity for the vacuum. This is a consequence of our choice of an absorption mechanism for the detector. Had we chosen a detector working on a stimulated emission principle, problems would arise with vacuum fluctuations. More generally the correlation between the field at the space-time point $x = (\boldsymbol{r},t)$ and the field at the space-time point $x' = (\boldsymbol{r},t')$ may be written as the correlation function

$$G^{(1)}(x,\ x') = \mathrm{Tr}\{\rho E^{(-)}(x)E^{(+)}(x')\}\ . \tag{3.9}$$

The first-order correlation function of the radiation field is sufficient to account for classical interference experiments. To describe experiments involving intensity correlations such as the Hanbury-Brown and Twiss experiment, it is necessary to define higher-order correlation functions. We define the nth-order correlation function of the electromagnetic field as

$$G^{(n)}(x_1\ \dots\ x_n,\ x_{n+1}\ \dots\ x_{2n}) = \mathrm{Tr}\{\rho E^{(-)}(x_1)\ \dots\ E^{(-)}(x_n) \times E^{(+)}(x_{n+1})\ \dots\ E^{(+)}(x_{2n})\}\,. \tag{3.10}$$

Such an expression follows from a consideration of an n-atom photon detector [1]. The n-fold delayed coincidence rate is

$$W^{(n)}(t_1 \dots\ t_n) = s^n G^{(n)}(r_1t_1\ \dots\ r_nt_n,\ r_nt_n\ \dots\ r_1t_1)\,, \tag{3.11}$$

where s is the sensitivity of the detector.

3.2 Properties of the Correlation Functions

A number of interesting inequalities can be derived from the general expression

$$\mathrm{Tr}\{\rho A^\dagger A\} \geq 0\,, \tag{3.12}$$

which follows from the non-negative character of $A^\dagger A$ for any linear operator A.

Thus choosing $A = E^{(+)}(x)$ gives

$$G^{(1)}(x,\ x) \geq 0\,. \tag{3.13}$$

In general, taking

$$A = E^{(+)}(x_n)\ \dots\ E^{(+)}(x_1) \tag{3.14}$$

yields

$$G^{(n)}(x_1\ \dots\ x_n,\ x_n\ \dots\ x_1) \geq 0 \tag{3.15}$$

Choosing

$$A = \sum_{j=1}^{n} \lambda_j E^{(+)}(x_j)\,, \tag{3.16}$$

where λ_j are an arbitrary set of complex numbers gives

$$\sum_{ij} \lambda_i^* \lambda_j G^{(1)}(x_i,x_j) \geq 0\,. \tag{3.17}$$

Thus the set of correlation functions $G^{(1)}(x_i,x_j)$ forms a matrix of coefficients for a positive definite quadratic form. Such a matrix has a positive determinant, i.e.,

$$\det[G^{(1)}(x_i,x_j)] \geq 0\,. \tag{3.18}$$

For $n = 1$, this is simply (3.13). For $n = 2$ we find

$$G^{(1)}(x_1,x_1)G^{(1)}(x_2,x_2) \geq |G^{(1)}(x_1,x_2)|^2 \tag{3.19}$$

which is a simple generalisation of the Schwarz inequality.

Choosing

$$A = \lambda_1 E^{(+)}(x_1)\ \dots\ E^{(+)}(x_n) + \lambda_2 E^{(+)}(x_{n+1})\ \dots\ E^{(+)}(x_{2n})\ , \tag{3.20}$$

we find the general relation

$$\begin{aligned} &G^{(n)}(x_1\ \dots\ x_n, x_n\ \dots x_1) G^{(n)}(x_{n+1}\ \dots\ x_{2n}, x_{2n}\ \dots x_{n+1}) \\ &\quad \geq |G^{(n)}(x_1\ \dots\ x_n, x_{n+1}\ \dots\ x_{2n})|^2\ . \end{aligned} \tag{3.21}$$

For two beams we may take

$$A = \lambda_1 E_1^{(+)}(x) E_1^{(+)}(x') + \lambda_2 E_2^{(+)}(x) E_2^{(+)}(x')\ , \tag{3.22}$$

with $x \equiv (\boldsymbol{r}, 0)$ and $x' \equiv (\boldsymbol{r}, t)$. The Cauchy–Schwartz inequality then becomes

$$G_{11}^{(2)}(0) G_{22}^{(2)}(0) \geq [G_{12}^{(2)}(t)]^2\ , \tag{3.23}$$

where

$$G_{ij}^{(2)}(t) = \mathrm{Tr}\{\rho E_i^{(-)}(x) E_i^{(-)}(x') E_j^{(+)}(x') E_j^{(+)}(x)\}\ ; \tag{3.24}$$

we have noted explicitly that $G_{ii}^{(2)}$ is time independent.

An inequality closely related to (3.23) may be derived by choosing

$$A = \lambda_1 E_1^{(-)}(x) E_1^{(+)}(x) + \lambda_2 E_2^{(-)}(x) E_2^{(+)}(x)\ . \tag{3.25}$$

This gives

$$\begin{aligned} &|\langle E_1^{(-)}(x) E_1^{(+)}(x) E_2^{(+)}(x) E_2^{(-)}(x)\rangle|^2 \\ &\quad \leq \langle [E_1^{(-)}(x) E_1^{(+)}(x)]^2\rangle \langle [E_2^{(-)}(x) E_2^{(+)}(x)]^2\rangle\ . \end{aligned} \tag{3.26}$$

This inequality will be used in Chap. 5.

3.3 Correlation Functions and Optical Coherence

Classical optical interference experiments correspond to a measurement of the first-order correlation function. We shall consider Young's interference experiment as a measurement of the first-order correlation function of the field and show how a definition of first-order optical coherence arises from considerations of the fringe visibility.

A schematic sketch of Young's interference experiment is depicted in Fig. 3.1. The field incident on the screen at position $\boldsymbol{r}$ and time t is the superposition of the fields at the two pin holes

$$E^{(+)}(\boldsymbol{r},\ t) = E_1^{(+)}(\boldsymbol{r},\ t) + E_2^{(+)}(\boldsymbol{r},\ t) \tag{3.27}$$

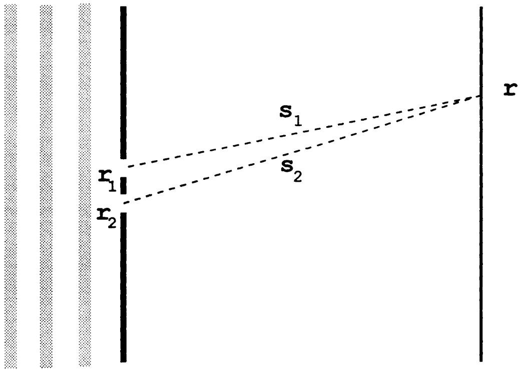

Fig. 3.1 Schematic representation of Young's interference experiment

where $E_i^{(+)}(\boldsymbol{r},t)$ is the field produced by pinhole i at the screen with

$$E_i^{(+)}(\boldsymbol{r},t) = E_i^{(+)}\left(\boldsymbol{r}_i,\, t - \frac{s_i}{c}\right)\left(\frac{1}{s_i}\right)\, \mathrm{e}^{\mathrm{i}(\mathrm{k}-\frac{\omega}{\mathrm{c}})\,\mathrm{s_i}} \tag{3.28}$$

where $s_i = |\boldsymbol{r}_i - \boldsymbol{r}|$

and $E_i^{(+)}(\boldsymbol{r}_i, t - s_i/c)$ is the field at the ith pinhole and for a spherical wave

$$k - \frac{\omega}{c} = 0\,.$$

Therefore (3.27) becomes

$$E^{(+)}(\boldsymbol{r},\, t) = \frac{E_1^{(+)}\left(\boldsymbol{r}_1,\, t - \frac{s_1}{c}\right)}{s_1} + \frac{E_2^{(+)}\left(\boldsymbol{r}_2,\, t - \frac{s_2}{c}\right)}{s_2}\,. \tag{3.29}$$

For $s_1 \approx s_2 \approx R$, we have

$$E^{(+)}(\boldsymbol{r},\, t) = \frac{1}{R}[E_1^{(+)}(x_1) + E_2^{(+)}(x_2)] \tag{3.30}$$

where

$$x_1 = \left(\boldsymbol{r}_1,\, t - \frac{s_1}{c}\right), \qquad x_2 = \left(\boldsymbol{r}_2,\, t - \frac{s_2}{c}\right)\,.$$

The intensity observed on the screen is proportional to

$$I = \mathrm{Tr}\{\rho E^{(-)}(\boldsymbol{r},\, t) E^{(+)}(\boldsymbol{r},\, t)\}\,. \tag{3.31}$$

Using (3.27) we find

$$I = G^{(1)}(x_1,\, x_1) + G^{(1)}(x_2,\, x_2) + 2\mathrm{Re}\{G^{(1)}(x_1,\, x_2)\} \tag{3.32}$$

where the R^{-2} factor is absorbed into a normalisation constant.

The first two terms on the right-hand side are the intensities from each pinhole in the absence of the other. The third term is the interference term. The correlation function for $x_1 \neq x_2$, in general takes on complex values. Writing this as

$$G^{(1)}(x_1,\, x_2) = |G^{(1)}(x_1,\, x_2)|\, \mathrm{e}^{\mathrm{i}\Psi(x_1,\, x_2)}\,, \tag{3.33}$$

we find

$$I = G^{(1)}(x_1,\, x_1) + G^{(1)}(x_2,\, x_2) + 2|G^{(1)}(x_1,\, x_2)| \cos\Psi(x_1,\, x_2)\,. \tag{3.34}$$

The interference fringes arise from the oscillations of the cosine term. The envelope of the fringes is described by the correlation function $G^{(1)}(x_1,x_2)$.

3.4 First-Order Optical Coherence

The idea of coherence in optics was first associated with the possibility of producing interference fringes when two fields are superposed. The highest degree of optical coherence was associated with a field which exhibits fringes with maximum visibility. If $G^{(1)}(x_1,x_2)$ was zero there would be no fringes and the fields are then described as incoherent. Thus the larger $G^{(1)}(x_1,x_2)$ the more coherent the field. The magnitude of $|G^{(1)}(x_1,x_2)|$ is limited by the relation

$$|G^{(1)}(x_1,\, x_2)| \leq [G^{(1)}(x_1,\, x_1) G^{(1)}(x_2,\, x_2)]^{1/2}\,. \tag{3.35}$$

The best possible fringe contrast is given by the equality sign. Thus the necessary condition for full coherence is

$$|G^{(1)}(x_1,\, x_2)| = [G^{(1)}(x_1,\, x_1) G^{(1)}(x_2,\, x_2)]^{1/2}\,. \tag{3.36}$$

Introducing the normalized correlation function

$$g^{(1)}(x_1,\, x_2) = \frac{G^{(1)}(x_1,x_2)}{[G^{(1)}(x_1,x_1) G^{(1)}(x_2,x_2)]^{1/2}}\,, \tag{3.37}$$

the condition (3.36) becomes

$$|g^{(1)}(x_1,\, x_2)| = 1 \tag{3.38}$$

or

$$g^{(1)}(x_1,\, x_2) = \mathrm{e}^{\mathrm{i}\Psi(x_1,\, x_2)}\,.$$

The visibility of the fringes is given by

$$\upsilon = \frac{I_{\max} - I_{\min}}{I_{\max} - I_{\min}}\,. \tag{3.39}$$

Using (3.27 and 3.31) for the intensity we may write υ as

$$\begin{aligned}\upsilon &= \left|\frac{G^{(1)}(x_1,\, x_2)}{(G^{(1)}(x_1,\, x_1)G^{(1)}(x_2,\, x_2))^{1/2}}\right| \frac{2(I_1 I_2)^{1/2}}{I_1+I_2} \\ &= |g^{(1)}| \frac{2(I_1 I_2)^{1/2}}{I_1+I_2} \,. \end{aligned} \tag{3.40}$$

If the fields incident on each pinhole have equal intensities the fringe visibility is equal to $|g^{(1)}|$. Thus the condition for first-order optical coherence $|g^{(1)}| = 1$ corresponds to the condition for maximum fringe visibility.

A more general definition of first-order coherence of the field $E(x)$ is that the first-order correlation function factorizes

$$G^{(1)}(x_1,\, x_2) = \varepsilon^{(-)}(x_1)\varepsilon^{(+)}(x_2)\,. \tag{3.41}$$

It is readily seen that this is equivalent to the condition for first-order optical coherence given by (3.38). It is clear that for a field in a left eigenstate of the operator $E^{(+)}(x)$ this factorization holds. The coherent states are an example of such a field. It is precisely this coherence property of the coherent states which led to their names.

We may generalize (3.41) to give the condition for nth optical coherence. This requires that the nth order correlation function factorizes:

$$G^{(n)}(x_1\,\ldots\, x_n, x_{n+1},\,\ldots\,, x_{2n}) = \varepsilon^{(-)}(x_1)\,\ldots\,\varepsilon^{(-)}(x_n)\varepsilon^{(+)}(x_{n+1})\,\ldots\,\varepsilon^{(-)}(x_{2n})\,. \tag{3.42}$$

Again the coherent states possess nth-order optical coherence.

Photon interference experiments of the kind typified by Young's interference experiment and Michelson's interferometer played a central role in early discussions of the dual wave and corpuscular nature of light. These experiments basically detect the interference pattern resulting from the superposition of two components of a light beam. Classical theory based on the wave nature of light readily explained the observed interference pattern. The quantum-mechanical explanation is based on the interference of the probability amplitudes for the photon to take either of two paths. We shall demonstrate how interference occurs even for a one photon field. For full details of the classical theory and experimental arrangements the reader is referred to the classic text of *Born* and *Wolf* [2].

We consider an interference experiment of the type performed by Young which consists of light from a monochromatic point source S incident on a screen possessing two pinholes P_1 and P_2 which are equidistant from S (see Fig. 3.1).

The pinholes act as secondary monochromatic point sources which are in phase and the beams from them are superimposed on a screen at position $\boldsymbol{r}$ and time t. In this region an interference pattern is formed.

To avoid calculating the diffraction pattern for the pinhole, we assume their dimensions are of the order of the wavelength of light in which case they effectively act as sources for single modes of spherical radiation in keeping with Huygen's principle. The appropriate mode functions for spherical radiation are

$$u_k(\mathbf{r}) = \sqrt{\frac{1}{4\pi L}}\frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}}{\mathbf{r}}\hat{e}_{\mathbf{k}}\,, \tag{3.43}$$

where L is the radius of the normalization volume, and $\hat{\boldsymbol{e}}_k$ is the unit polarization vector.

The field detected on the screen at position $\boldsymbol{r}$ and time t is then the sum of the two spherical modes emitted by the two pinholes

$$E^{(+)}(\mathbf{r},\,t) = f(\mathbf{r},t)(a_1\mathrm{e}^{\mathrm{i}ks_1} + a_2\mathrm{e}^{\mathrm{i}ks_2})\,, \tag{3.44}$$

with

$$f(\boldsymbol{r},\,t) = \mathrm{i}\left(\frac{\hbar\omega}{2}\right)^{1/2}\frac{\hat{e}_k}{(4\pi L)^{1/2}}\frac{1}{R}\mathrm{e}^{-\mathrm{i}\omega t},$$

where s_1 and s_2 are the distances of the pinholes P_1 and P_2 to the point on the screen, and we have set $s_1 \approx s_2 = R$ in the denominator of the mode functions. Substituting (3.43) into (3.2) for the intensity we find

$$I(\boldsymbol{r},\,t) = \eta[\mathrm{Tr}\{\rho a_1^\dagger a_1\} + \mathrm{Tr}\{\rho a_2^\dagger a_2\} + 2|\mathrm{Tr}\{\rho a_1^\dagger a_2\}|\cos\Phi]\,. \tag{3.45}$$

where

$$\begin{aligned}\mathrm{Tr}\{\rho a_1^\dagger a_2\} &= |\mathrm{Tr}\{\rho a_1^\dagger a_2\}|\mathrm{e}^{\mathrm{i}\phi},\\ \eta &= |f(\boldsymbol{r},\,t)|^2,\\ \Phi &= k(s_1 - s_2) + \phi.\end{aligned}$$

This expression exhibits the typical interference fringes with the maximum of intensity occurring at

$$k(s_1 - s_2) + \phi = n2\pi\,, \tag{3.46}$$

with n an integer.

The maximum intensity of the fringes falls off as one moves the point of observation further from the central line by the R^{-2} factor in $|f(\boldsymbol{r},t)|^2$.

We shall evaluate the intensity for fields which may be generated by a single-mode excitation and hence have first-order coherence. A general representation of such a field is

$$|\psi\rangle = f(b^\dagger)|0\rangle\,, \tag{3.47}$$

where $|0\rangle$ denotes the vacuum state of the radiation field and $b^\dagger$ is the creation operator for a single mode of the radiation field. The operator $b^\dagger$ may be expressed as a linear combination of $a_1^\dagger$ and $a_2^\dagger$ as follows

$$b^\dagger = -\frac{1}{\sqrt{2}}(a_1^\dagger + a_2^\dagger)\,, \tag{3.48}$$

where we have assumed equal intensities through each slit. We shall now consider as a special case the field with only one photon incident on the pinholes, i.e.,

$$|1\text{ photon}\rangle = b^\dagger|0\rangle = \frac{1}{\sqrt{2}}(|1,0\rangle + |0,1\rangle)\,, \tag{3.49}$$

where the notation used for the eigenkets $|n_1, n_2\rangle$ implies that there are n_1 photons present in mode k_1 and n_2 photons present in mode k_2. This state of the field reflects the fact that we don't know which pinhole the photon goes through.

From (3.45) this yields the following expression for the mean intensity on the screen

$$I(\boldsymbol{r},\, t) = \eta(1 + \cos\Phi)\ . \tag{3.50}$$

It is clear from this equation that an interference pattern may be built up from a succession of one-photon interference fringes.

The quantum explanation for the interference pattern was first put forward by *Dirac* [3] in his classic text on quantum mechanics. There he argued that the observed intensity pattern results from interference between the probability amplitudes of a single photon to take either of two possible paths. The crux of the quantum mechanical explanation is that the wavefunction gives information about the probability of one photon being in a particular place and not the probable number of photons in that place. *Dirac* pointed out that the interference between the two beams does not arise because photons of one beam sometimes annihilate photons from the other, and sometimes combine to produce four photons. "This would contradict the conservation of energy. The new theory which connects the wave functions with probabilities for one photon gets over the difficulty by making each photon go partly into each of two components. Each photon then interferes only with itself. Interference between two different photons never occurs". We stress that the above-quoted statement of *Dirac* was only intended to apply to experiments of the Young's type where the interference pattern is revealed by detecting single photons. It was not intended to apply to experiments of the type where correlations between different photons are measured.

A very early experiment to test if interference would result from a single photon was performed by *Taylor* [4] in 1905. In this experiment the intensity of the source was so low that on average only one photon was incident on the slits at a time. The photons were detected on a photographic plate so that the detection time was very large. Interference fringes were observed in this experiment. This experiment did not definitively show that the interference fringes resulted from a single photon since the statistical distribution of photons meant that sometimes two photons could be incident on the slits. A definitive experiment was conducted by *Grangier* et al. [5] using a two-photon cascade as a source. A coincidence technique which detected one photon of the pair enabled them to prepare a one photon source.

We now consider the interference patterns produced by other choices of a field.

3.5 Coherent Field

We consider a coherent field as generated by an ideal laser incident on the pinholes. The wavefunction for this coherent field is

$$|\text{coherent field}\rangle = |\alpha_1,\ \alpha_2\rangle = |\alpha_1\rangle|\alpha_2\rangle\ . \tag{3.51}$$

Since this wavefunction is a product state, it may well represent two independent light beams. This particular product may, however, be generated by a single-mode excitation in the following manner:

$$\begin{aligned}|\alpha_1\rangle|\alpha_2\rangle &= \exp(\alpha b^\dagger - \alpha^* b)|0\rangle \\ &= \exp\frac{1}{\sqrt{2}}(\alpha a_1^\dagger - \alpha^* a_1)\exp\frac{1}{\sqrt{2}}(\alpha a_2^\dagger - \alpha^* a_2)|0\rangle \\ &= \left|\frac{\alpha}{\sqrt{2}}\right\rangle\left|\frac{\alpha}{\sqrt{2}}\right\rangle .\end{aligned} \tag{3.52}$$

The intensity pattern produced by this coherent field is

$$I(\boldsymbol{r},t) = \eta(|\alpha|^2 + |\alpha|^2 \cos\phi) . \tag{3.53}$$

The above example demonstrates the possibility of obtaining interference between independent light beams. Experimentally, this requires that the phase relation between the two beams be slowly varying or else the fringe pattern will be washed out. Such experiments have been performed by *Pfleegor* and *Mandel* [6]. Interference between independent light beams is, however, only possible for certain states of the radiation field, for example, the coherent states as demonstrated above. Interference is not generally obtained from independent light beams, as we shall demonstrate in the following example. We consider the two independent light beams to be Fock states, that is, described by the wavefunction

$$|\psi\rangle = |n_1\rangle|n_2\rangle . \tag{3.54}$$

This yields a zero correlation function and consequently no fringes are obtained.

The analysis we performed leading to (3.50) bears out *Dirac's* argument that the interference fringes may be produced by a series of one photon experiments. However, Young's interference fringes may perfectly well be explained by the interference of classical waves. Experiments of this kind which measure the first-order correlation functions of the electromagnetic field do not distinguish between the quantum and classical theories of light.

3.6 Photon Correlation Measurements

The first experiment performed outside the domain of one photon optics was the intensity correlation experiment of *Hanbury-Brown* and *Twiss* [7]. Although the original experiment involved the analogue correlation of photo-currents, later experiments used photon counters and digital correlations and were truly photon correlation measurements. In essence these experiments measure the joint photocount probability of detecting the arrival of a photon at time t and another photon at time $t+\tau$. This may be written as an intensity or photon-number correlation function.

Using the quantum detection theory developed by *Glauber*, the measured quantity is the normally ordered correlation function

$$\begin{aligned} G^{(2)}(\tau) &= \langle E^{(-)}(t)E^{(-)}(t+\tau)E^{(+)}(t+\tau)E^{(+)}(t)\rangle \\ &= \langle :\ I(t)I(t+\tau)\ :\rangle \\ &\propto \langle :\ n(t)n(t+\tau)\ :\rangle \end{aligned} \tag{3.55}$$

where : : indicates normal ordering, $I(t)$ is the intensity for analogue measurements and $n(t)$ is the photon number in photon counting experiments. It is useful to introduce the normalized second-order correlation function defined by

$$g^{(2)}(\tau) = \frac{G^{(2)}(\tau)}{|G^{(1)}(0)|^2}\,. \tag{3.56}$$

We shall evaluate $g^{(2)}(\tau)$ for certain classes of field. For a field which possesses second-order coherence

$$G^{(2)}(\tau) = \varepsilon^{(-)}(t)\varepsilon^{(-)}(t+\tau)\varepsilon^{(+)}(t+\tau)\varepsilon^{(+)}(t) = [G^{(1)}(0)]^2\,. \tag{3.57}$$

Hence $g^{(2)}(\tau) = 1$.

For a fluctuating classical field we may introduce a probability distribution $P(\varepsilon)$ describing the probability of the field $E^{(+)}(\varepsilon,t)$ having the amplitude ε where

$$E^{(+)}(\varepsilon,\,t) = -\left(\mathrm{i}\frac{\hbar\omega}{2\varepsilon_0\,V}\right)^{1/2}\,\varepsilon\mathrm{e}^{-\mathrm{i}\omega t}\,.$$

For a multimode field we have a multivariate probability distribution $P(\{\varepsilon_k\})$. The second-order correlation function $G^{(2)}(\tau)$ may be written as

$$G^{(2)}(\tau) = \int P(\{\varepsilon_k\})E^{(-)}(\varepsilon_k,\,t)E^{(-)}(\varepsilon_k,\,t+\tau)E^{(+)}(\varepsilon_k,\,t+\tau)E^{(+)}(\varepsilon_k,\,t)d^2\{\varepsilon_k\}\,. \tag{3.58}$$

For zero time delay $\tau = 0$ we may write for a single-mode field

$$g^{(2)}(0) = 1 + \frac{\int P(\varepsilon)(|\varepsilon|^2 - \langle|\varepsilon|^2\rangle)^2 d^2\varepsilon}{(\langle|\varepsilon|^2\rangle)^2}\,. \tag{3.59}$$

For classical fields the probability distribution $P(\varepsilon)$ is positive, hence $g^{(2)}(0) \geq 1$.

For a field obeying Gaussian statistics with zero mean amplitude

$$\begin{aligned} &\langle E^{(-)}(\varepsilon,\,t)E^{(-)}(\varepsilon,\,t+\tau)E^{(+)}(\varepsilon,\,t)E^{(+)}(\varepsilon,\,t+\tau)\rangle \\ &\quad= \langle E^{(-)}(\varepsilon,\,t)E^{(-)}(\varepsilon,\,t+\tau)\rangle\langle E^{(+)}(\varepsilon,\,t+\tau)E^{(+)}(\varepsilon,\,t)\rangle \\ &\qquad+ \langle E^{(-)}(\varepsilon,\,t)E^{(+)}(\varepsilon,\,t)\rangle\langle E^{(-)}(\varepsilon,\,t+\tau)E^{(+)}(\varepsilon,\,t+\tau)\rangle \\ &\qquad+ \langle E^{(-)}(\varepsilon,\,t)E^{(+)}(\varepsilon,\,t+\tau)\rangle\langle E^{(-)}(\varepsilon,\,t+\tau)E^{(+)}(\varepsilon,\,t)\rangle\,. \end{aligned} \tag{3.60}$$

For fields with no phase-dependent fluctuations the first term may be neglected. Then

$$G^{(2)}(\tau) = G^{(1)}(0)^2 + |G^{(1)}(\tau)|^2 \,. \tag{3.61}$$

Hence the normalized second-order correlation function is

$$g^{(2)}(\tau) = 1 + |g^{(1)}(\tau)|^2 \,. \tag{3.62}$$

Now $G^{(1)}(\tau)$ is the Fourier transform of the spectrum of the field

$$S(\omega) = \int_{-\infty}^{\infty} \mathrm{d}\tau \, \mathrm{e}^{-\mathrm{i}\omega\tau} G^{(1)}(\tau) \,. \tag{3.63}$$

Hence for a field with a Lorentzian spectrum

$$g^{(2)}(\tau) = 1 + \mathrm{e}^{-\gamma\tau} \tag{3.64}$$

and for a field with a Gaussian spectrum

$$g^{(2)}(\tau) = 1 + \mathrm{e}^{-\gamma^2\tau^2} \,, \tag{3.65}$$

where γ is the spectral linewidth.

For a values of $\tau \gg \tau_c$ the correlation time of the light, the correlation function factorizes and $g^{(2)}(\tau) \to 1$. The increased value of $g^{(2)}(\tau)$ for $\tau < \tau_c$ for chaotic light over coherent light $[g^{(2)}(0)_{\text{chaotic}} = 2g^{(2)}(0)_{\text{coh}}]$ is due to the increased intensity fluctuations in the chaotic light field. There is a high probability that the photon which triggers the counter occurs during a high intensity fluctuation and hence a high probability that a second photon will be detected arbitrarily soon. This effect known as photon bunching was first detected by *Hanbury-Brown* and *Twiss*. Later experiments [8] showed excellent agreement with the theoretical predictions for chaotic and coherent light (Fig. 3.2). We note that the above analysis does not rely on any quantisation of the electromagnetic field but may be deduced from a purely classical analysis of the electromagnetic field with fluctuating amplitudes for the modes.

Measurement of the second-order correlation function of light with Gaussian statistics has formed the basis of photon correlation spectroscopy [9]. Photon correlation spectroscopy may be used to measure very narrow linewidths (1–10^8 Hz) which are outside the range of conventional spectrometers. The second-order correlation function $g^2(\tau)$ is measured using electronic correlators and the linewidth extracted using (3.64 or 3.65). This has found application, for example, in the measurement of the diffusion coefficient of macromolecules where the scattered light has Gaussian statistics. The linewidth of the scattered light contains information on the diffusion coefficient of the macromolecule. This technique has been applied to determine the size of biological molecules such as viruses as well as in studying turbulent flows.

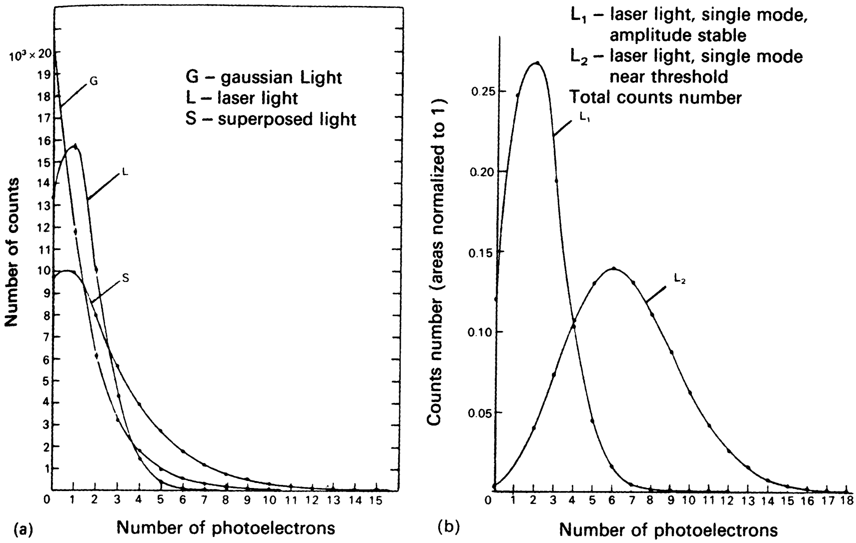

Fig. 3.2 Measured photo-count statistics for (**a**) Gaussian, laser and superposed fields. Measuring time of a single sample: 10 μs. Coherence time of the Gaussian field; 40 μs. (**b**) Two laser fields. Measuring time of a single sample: 10 μs

3.7 Quantum Mechanical Fields

We shall now evaluate the second-order correlation function for some quantum-mechanical fields. We shall restrict our attention to a single-mode field and calculate $g^{(2)}(0)$ and the variance in the photon number $V(n)$

$$g^{(2)}(0) = \frac{\langle a^\dagger a^\dagger a a\rangle}{\langle a^\dagger a\rangle^2} = 1 + \frac{V(n) - \bar{n}}{\bar{n}^2}, \tag{3.66}$$

where $V(n) = \langle (a^\dagger a)^2\rangle - \langle a^\dagger a\rangle^2$.

Coherent State

For a coherent state

$$\rho = |\alpha\rangle\langle\alpha|, \quad g^{(2)}(0) = 1 \tag{3.67}$$

and $V(n) = \bar{n}$ for a Poisson distribution in photon number.

Number state

$$\rho = |n\rangle\langle n|, \qquad g^{(2)}(0) = 1 - \frac{1}{n}, \quad n > 2\,. \tag{3.68}$$

A number state has zero variance in the photon number ($V(n) = 0$). If $g^{(2)}(\tau) < g^{(2)}(0)$ there is a tendency for photons to arrive in pairs. This situation is referred to as *photon bunching*. The converse situation, $g^{(2)}(\tau) > g^{(2)}(0)$ is called *antibunching*. As noted above, however, $g^{(2)}(\tau) \to 1$ on a sufficiently long time scale. Thus a field for which $g^{(2)}(0) < 1$ will always exhibit antibunching on some time scale.

A value of $g^{(2)}(0)$ less than unity could not have been predicted by a classical analysis. Equation (3.59) always predicts $g^{(2)}(0) \geq 1$. To obtain a $g^{(2)}(0) < 1$ would require the field to have elements of negative probability, which is forbidden for a true probability distribution. This effect known as photon antibunching is a feature peculiar to the quantum mechanical nature of the electromagnetic field.

A distinction should be maintained between photon antibunching and sub-Poissonian statistics, although the two phenomena are closely related. For Poisson statistics the variance of the photon number is equal to the mean. Thus a measure of sub-Poissonian statistics is provided by the quantity $V(N) - \langle N \rangle$. For a stationary field one may show that [10].

$$V(N) - \langle N \rangle = \frac{\langle N \rangle^2}{T^2} \int_{-T}^{T} \mathrm{d}\tau (T - |\tau|)[g^{(2)}(\tau) - 1] , \tag{3.69}$$

where T is the counting time interval. If $g^{(2)}(\tau) = 1$ the field exhibits Poisson statistics. Certainly a field for which $g^{(2)}(\tau) < 1$ for all τ will exhibit sub-Poissonian statistics. However, it is possible to specify fields for which $g^{(2)}(\tau) > g^{(2)}(0)$ but which exhibit super-Poissonian statistics over some time interval.

3.7.1 Squeezed State

We consider a squeezed state $|\alpha, r\rangle$ with r defined as positive (Fig. 3.3). We align our axes such that the X_1 direction is parallel to the minor axis of the error ellipse. The direction (*1*) is referred to as the direction of squeezing and the direction (*2*) as the direction of coherent excitation. We then define α by $2\alpha = \langle X_1 \rangle + \mathrm{i}\langle X_2 \rangle$ with $\theta = \tan^{-1} (\langle X_2 \rangle / \langle X_1 \rangle)$. The variance in the photon number for this squeezed state is

$$\frac{V(n) - \bar{n}}{\bar{n}^2} = \frac{|\alpha|^2(\cosh 2r - \sinh 2r \cos 2\theta - 1) + \sinh^2 r \cosh 2r}{(|\alpha|^2 + \sinh^2 r)^2} . \tag{3.70}$$

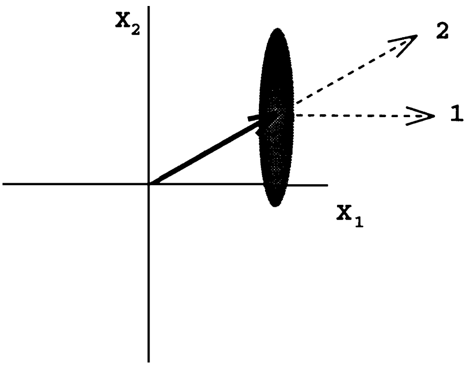

Fig. 3.3 A phase convention for squeezed states. Direction *1* is the direction of squeezing, direction *2* is the direction of coherent excitation. The error ellipse is aligned so that the squeezing direction is parallel to the X_1 direction

When $\theta = \pi/2$, that is the squeezing is out of phase with the complex amplitude

$$V(n) = |\alpha|^2 e^{2r} + 2\sinh^2 r \cosh^2 r \,. \tag{3.71}$$

Thus this state with increased amplitude fluctuations has super-Poissonian statistics as expected.

When $\theta = 0$, that is the squeezing is in phase with the complex amplitude

$$V(n) = |\alpha|^2 e^{-2r} + 2\sinh^2 r \cosh^2 r \,. \tag{3.72}$$

The first term corresponds to the reduction in number fluctuations in the original Poisson distribution. The second term is due to the fluctuations of the additional photons in the squeezed vacuum.

When $|\alpha|^2 \gg 2\sinh^2 r \cosh^2 r$ this is an amplitude squeezed state with sub-Poissonian photon statistics. The maximum reduction in photon number fluctuations one can get in an amplitude squeezed state may be estimated as follows: For $r \geq 1$

$$V(n) \approx |\alpha|^2 e^{-2r} + \frac{1}{8} e^{4r} \,. \tag{3.73}$$

The minimum value of $V(n)$ occurs for $e^{6r} = 4|\alpha|^2$ which corresponds to $V_{\min}(n) \approx 0.94|\alpha|^{4/3}$. Diagrams depicting squeezed states with reduced amplitude and reduced phase fluctuations are shown in Fig. 3.4.

In Chap. 5 we will discuss a nonlinear interaction which produces a state with Poisson distribution in photon number, but can also exhibit amplitude squeezing.

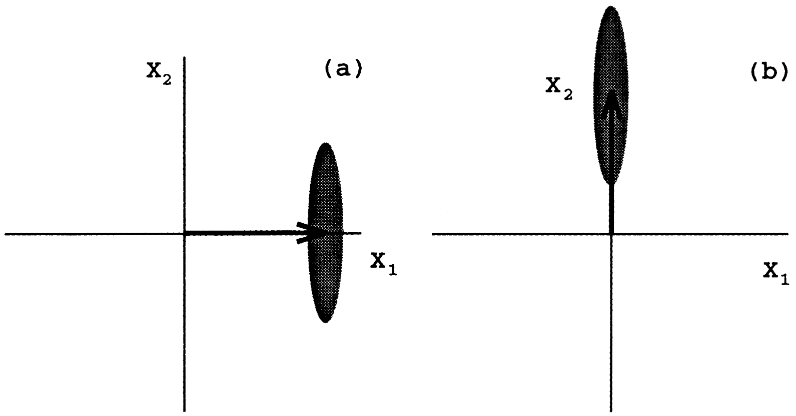

Fig. 3.4 Phase-space of amplitude and phase squeezed states. (**a**) The quadrature carrying the coherent excitation is squeezed $(\theta = 0)$. (**b**) The quadrature out of phase with the coherent excitation is squeezed $(\theta = \pi/2)$

3.7.2 Squeezed Vacuum

For a squeezed vacuum $\alpha = 0$

$$V(n) = \bar{n}(1 + \cosh 2r) . \tag{3.74}$$

Hence a squeezed vacuum always exhibits super-Poissonian statistics.

We may compare the characteristics of a squeezed state with that of a number state. A number state has reduced photon number fluctuations but has complete uncertainty in phase. Thus a number state will not show any squeezing. For a number state

$$\Delta X_1^2 = \Delta X_2^2 = 2n + 1 . \tag{3.75}$$

A number state may be represented in an (X_1, X_2) phase space plot as an annulus with radius $\sqrt{n}$ and width $= 1$.

3.8 Phase-Dependent Correlation Functions

The even-ordered correlation functions such as the second-order correlation function $G^{(n,n)}(x)$ contain no phase information and are a measure of the fluctuations in the photon number. The odd-ordered correlation functions $G^{(n,m)}$ $(x_1 \ldots x_n, x_{n+1} \ldots x_{n+m})$ with $n \neq m$ will contain information about the phase fluctuations of the electromagnetic field. The variances in the quadrature phases ΔX_1^2 and ΔX_2^2 are given by measurements of this type. A number of schemes to make quadrature phase measurements have been discussed by *Yuen* and *Shapiro* [11].

These schemes involve homodyning the signal field with a reference signal known as the local oscillator before photodetection. Homodyning with a reference signal of fixed phase gives the phase sensitivity necessary to yield the quadrature variances.

Consider two fields $E_1(\boldsymbol{r},t)$ and $E_2(\boldsymbol{r},t)$ of the same frequency, combined on a beam splitter with transmittivity η, as shown in Fig. 3.5. This configuration is essentially identical to the single field quadrature homodyne detection scheme discussed by *Yuen* and *Shapiro*.

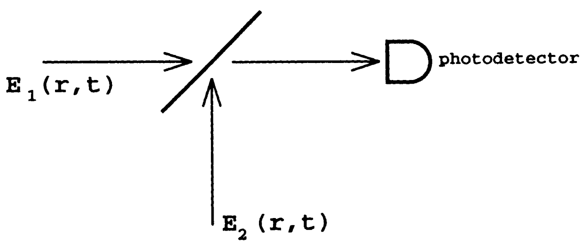

Fig. 3.5 Schematic representation of homodyne detection of squeezed states

We expand the two incident fields into the usual positive and negative frequency components

$$E_1(\boldsymbol{r},\, t) = \mathrm{i}\left(\frac{\hbar\omega}{2V\varepsilon_0}\right)^{1/2} \left(a\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot r-\omega t)} - a^\dagger \mathrm{e}^{-\mathrm{i}(\boldsymbol{k}\cdot r-\omega t)}\right), \tag{3.76}$$

$$E_2(\boldsymbol{r},\, t) = \mathrm{i}\left(\frac{\hbar\omega}{2V\varepsilon_0}\right)^{1/2} \left(b\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot r-\omega t)} - b^\dagger \mathrm{e}^{-\mathrm{i}(\boldsymbol{k}\cdot r-\omega t)}\right), \tag{3.77}$$

where a, b are boson operators which characterise the two modes E_1 and E_2, respectively. Both fields are taken to have the same sense of polarization, and are phase locked.

The total field after combination is given by

$$E_T(\boldsymbol{r},\, t) = \mathrm{i}\left(\frac{\hbar\omega}{2V\varepsilon_0}\right)^{1/2} \left(c\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot r-\omega t)} - c^\dagger \mathrm{e}^{-\mathrm{i}(\boldsymbol{k}\cdot r-\omega t)}\right), \tag{3.78}$$

where

$$c = \sqrt{\eta a} + \mathrm{i}\sqrt{1-\eta}\, b\,. \tag{3.79}$$

We have included a 90° phase shift between the reflected and transmitted beams at the beam splitter.

The photon detector, of course, responds to the moments of $c^\dagger c$. We thus define the number operator $\hat{N} = c^\dagger c$.

The mean photo-electron current in the detector is proportional to $\langle c^\dagger c\rangle$ which is given by

$$\langle c^\dagger c\rangle = \eta\langle a^\dagger a\rangle + (1-\eta)\langle b^\dagger b\rangle - \mathrm{i}\sqrt{\eta(1-\eta)}(\langle a\rangle\langle b^\dagger\rangle - \langle a^\dagger\rangle\langle b\rangle)\,. \tag{3.80}$$

Let us take the field E_2 to be the local oscillator and assume it to be in a coherent state of large amplitude β. Then we may neglect the first term in (3.80) and write $\langle c^\dagger c\rangle$ in the form

$$\langle c^\dagger c\rangle \approx (1-\eta)|\beta|^2 + |\beta|\sqrt{\eta(1-\eta)}\langle X_{\theta+\pi/2}\rangle\,, \tag{3.81}$$

where

$$X_\theta \equiv a\mathrm{e}^{-\mathrm{i}\theta} + a^\dagger \mathrm{e}^{\mathrm{i}\theta}\,, \tag{3.82}$$

and θ is the phase of β. We see that when the contribution from the reflected local-oscillator intensity level is subtracted, the mean photo-current in the detector is proportional to the mean quadrature phase amplitude of the signal field defined with respect to the local oscillator phase. If we change θ through $\pi/2$ we can determine the mean amplitude of the two canonically conjugate quadrature phase operators.

We now turn to a consideration of the fluctuations in the photo-current. The rms fluctuation current is determined by the variance of $c^\dagger c$. For an intense local oscillator in a coherent state this variance is

$$V(n_c) \approx (1-\eta)^2|\beta|^2 + |\beta|^2\eta(1-\eta)V(x_{\theta+\pi/2})\,. \tag{3.83}$$

The first term here represents reflected local oscillator intensity fluctuations. If this term is subtracted out, the photo-current fluctuations are determined by the variances in $X_{\theta+\pi/2}$, the measured quadrature phase operator. To subtract out the contribution of the reflected local oscillator field balanced homodyne detection may be used. In this scheme the output from both ports of the beam splitter is directed to a photodetector and the resulting currents combined with appropriate phase shifts before subsequent analysis. Balanced homodyne detection realises a direct measurement of the signal field quadrature phase operators [11]. Alternatively, the contribution from the local oscillator intensity fluctuations may be reduced by making the transmittivity $\eta \approx 1$, in which case the dominant contribution to $V(n_c)$ comes from the second term in (3.83).

3.9 Photon Counting Measurements

3.9.1 Classical Theory

Consider radiation of intensity $I(t)$ falling on a photo-electric counter. The probability that a count occurs in a time dt is given by

$$\Delta p(t) = \alpha I(t)\mathrm{d}t \, . \tag{3.84}$$

The parameter α is a measure of the sensitivity of the detector, and depends on the area of the detector and the spectral range of the incident light. Suppose initially there are no random fluctuations in the intensity $I(t)$. Now $1-\Delta p(t')$ represents the probability that no counts occur in the time interval dt' at t'. Then assuming the independence of photocounts in different time intervals the joint probability that no counts occur in an entire interval t to $t+T$ is given by the product

$$\begin{aligned}\prod_t^{t+T}[1-\Delta p(t')] &\approx \prod_t^{t+T}\exp[-\Delta p(t')] \\ &= \exp\left[-\sum_t^{t+T}\Delta p(t')\right] \\ &= \exp\left[-\int_t^{t+T}\mathrm{d}p(t')\right] \, . \end{aligned} \tag{3.85}$$

Thus the probability for no counts in the interval t to $t+T$ is

$$P_0(T+t,t) = \exp\left[-\alpha\int_t^{t+T} I(t')\mathrm{d}t'\right] \, . \tag{3.86}$$

The probability $P_1(T+t,t)$ that one count occurs between t and $t+T$ is

$$\sum_{t''} \mathrm{d}p(t'') \prod_{t}^{t+T} [1-\Delta p(t')] \to \int_{t}^{t+T} \mathrm{d}p(t'') \exp\left[-\int_{t}^{t+T} \mathrm{d}p(t')\right] . \tag{3.87}$$

Hence

$$P_1(T+t,\, t) = \left[\alpha \int_{t}^{t+T} I(t')\mathrm{d}t'\right] \exp\left[-\alpha \int_{t}^{t+T} I(t')\mathrm{d}t'\right] . \tag{3.88}$$

Following this reasoning the probability for n counts in the interval t to $t+T$ is

$$P_n(t,\, T) = \frac{1}{n!}[\alpha T \bar{I}(t,\, T)]^n \exp[-\alpha T \bar{I}(t,\, T)] , \tag{3.89}$$

where

$$\bar{I}(t,\, T) = \frac{1}{T} \int_{t}^{t+T} I(t')\mathrm{d}t'$$

is the mean intensity during the counting interval.

Now since $\bar{I}(t,T)$ may vary from one counting interval to the next, $P_n(T)$ is a time average of $P_n(t,T)$ over a large number of different starting times

$$\begin{aligned} P_n(T) &= \langle P_n(t,T) \rangle \\ &= \left\langle \frac{[\alpha \bar{I}(t,\, T)T]^n}{n!} \exp[-\alpha \bar{I}(t,\, T)T] \right\rangle . \end{aligned} \tag{3.90}$$

This formula was first derived by *Mandel* [12].

We note a useful generating function for the photon-counting distribution is

$$Q(\lambda,\, T) = \sum_{n=0}^{\infty} (1-\lambda)^n P_n(T) . \tag{3.91}$$

The factorial moments of the photon counting distribution may be obtained as follows:

$$\begin{aligned} \overline{n(n-1)\dots(n-k)} &= \sum_{n=0}^{\infty} n(n-1)\dots(n-k) P_n(T) \\ &= (-1)^k \frac{\partial^k}{\partial \lambda^k} Q(\lambda,\, T)\bigg|_{\lambda=0} . \end{aligned} \tag{3.92}$$

We shall now consider some important cases of the photon counting formula (3.89).

3.9.2 Constant Intensity

In the simplest case of a constant intensity $\bar{I}(t,T)$ is independent of t and T, hence

$$\bar{I}(t,\ T) = I\ . \tag{3.93}$$

In this case the averaging over a fluctuating intensity $I(t)$ is unnecessary and

$$P_n(T) = \frac{\bar{n}^n}{n!}\exp(-\bar{n})\ , \tag{3.94}$$

where

$$\bar{n} = \alpha IT\ .$$

This is a Poisson distribution for which the variance $V(n) = \bar{n}$.

3.9.3 Fluctuating Intensity–Short-Time Limit

When the intensity is fluctuating, (3.89) can be simplified in the limit where the counting time T is short compared to the coherence time τ_c over which the intensity changes. If, during the interval $T, I(t)$ remains reasonably constant then

$$\bar{I}(t,T) = \bar{I}(t)\ . \tag{3.95}$$

With ergodic hypothesis for a stationary light source we may convert the time average in (3.90) into an ensemble average over the distribution $p(\bar{I}(t))$.

The photon counting formula may then be written

$$P_n(T) = \int_0^\infty \frac{[\alpha\bar{I}(t)T]^n}{n!}\mathrm{e}^{-\alpha I(t)T}p(\bar{I}(t))\mathrm{d}\bar{I}(t)\ . \tag{3.96}$$

In the following we replace $\bar{I}(t)$ by the stochastic variable I for ease of notation. The mean photon count is

$$\begin{aligned}\bar{n} = \sum_{n=0}^\infty nP_n(T) &= \int_0^\infty \sum_{n=0}^\infty n\frac{(\alpha IT)^n}{n!}\mathrm{e}^{-\alpha IT}p(\mathrm{I})\mathrm{d}I \\ &= \int_0^\infty \alpha TIp(I)\mathrm{d}I = \alpha T\langle I\rangle\ . \end{aligned}\tag{3.97}$$

Defining moments of intensity as

$$\langle I^n\rangle = \int_0^\infty I^n p(I)\mathrm{d}I\ , \tag{3.98}$$

we find for the mean square count

$$\overline{n^2} = \sum_{n=0}^{\infty} n^2 P_n(T) = \int_0^{\infty} (\alpha^2 T^2 I^2 + \alpha T I) p(I) \mathrm{d}I$$
$$= \alpha^2 T^2 \langle I^2 \rangle + \alpha T \langle I \rangle \,. \tag{3.99}$$

Thus the variance is

$$V(n) = \overline{n^2} - \bar{n}^2 = \alpha T \langle I \rangle + \alpha^2 T^2 (\langle I^2 \rangle - \langle I \rangle^2) \,. \tag{3.100}$$

We note that this is always greater than the mean unless $p(I)$ is a Dirac delta function $\delta(I - I_0)$. This is true for classical fields. For certain quantum mechanical fields we shall see that it is possible to obtain $V(n) < \bar{n}$.

A thermal light field has the following probability distribution for its intensity

$$p(I) = \frac{1}{I_0} \exp\left(\frac{-I}{I_0}\right) , \tag{3.101}$$

with moments

$$\langle I^n \rangle = n! I_0^n \,.$$

The mean and variance of the photocount distribution are

$$\bar{n} = \alpha T I_0, \quad V(n) = \bar{n}(1 + \bar{n}) \,. \tag{3.102}$$

The photon-counting distribution is

$$P_n(T) = \frac{(\alpha T)^n}{I_0 n!} \int_0^{\infty} I^n \exp\left[-I\left(\alpha T + \frac{1}{I_0}\right)\right] \mathrm{d}I$$
$$= \frac{(\alpha T)^n}{I_0 n!} \left(\alpha T + \frac{1}{I_0}\right)^{-(n+1)} \int_0^{\infty} x^n \mathrm{e}^{-x} \mathrm{d}x \, n!$$
$$= \frac{1}{(1+\bar{n})} \left(\frac{\bar{n}}{1+\bar{n}}\right)^n \,. \tag{3.103}$$

This power-law distribution for thermal light has been verified in photon-counting experiments. Experiments have also shown that the photon count distribution of highly stabilized lasers is approximated by a Poisson distribution [8, 13].

We conclude with a comment on the form assumed by $\bar{I}(t,T)$ if the depletion of the signal field by the detection process is taken into account. Then

$$I(t) = I_0 \mathrm{e}^{-\lambda t} \,, \tag{3.104}$$

where λ is the rate of photon absorption. Then

$$\bar{I}(t,\ T) = \frac{I_0}{T} \int_t^{t+T} \mathrm{e}^{-\lambda t'}\ \mathrm{d}t' \,, \tag{3.105}$$

Thus

$$\bar{I}(\tau,\ T) = \frac{I(t)}{\lambda T}(1 - \mathrm{e}^{-\lambda T}) \,. \tag{3.106}$$

We note that for short counting times this has the same form as (3.95).

3.10 Quantum Mechanical Photon Count Distribution

The photon count distribution for a quantum mechanical field may be written in a formally similar way to the classical expression [14]

$$P_n(T) = \left\langle : \frac{[\alpha \bar{I}(T)T]^n}{n!} \exp[-\alpha \bar{I}(T)T] : \right\rangle \tag{3.107}$$

where

$$\begin{aligned} \bar{I}(T) &= \frac{1}{T} \int_0^T I(t)\mathrm{d}t \\ &= \frac{1}{T} \int_0^T E^{(-)}(\boldsymbol{r},\ t) E^{(+)}(\boldsymbol{r},\ t)\ \mathrm{d}t \end{aligned} \tag{3.108}$$

and : : denotes normal ordering of the operators. We shall demonstrate the use of this formula for a single-mode field, in which case (3.107) may be written as

$$P_n(T) = \mathrm{Tr}\left(\rho : \frac{[\mu(T) a^\dagger a]^n}{n!} \exp[-a^\dagger a \mu(T)] :\right) \tag{3.109}$$

where $\mu(T)$ is the probability for detecting one photon in time T from a one photon field. The explicit form of $\mu(T)$ depends on the physical situation, e.g., $\mu(T) = \lambda T$ for an open system and $\mu(T) = (1 - \mathrm{e}^{-\lambda T})$ for a closed system.

The photon count distribution may be related to the diagonal matrix elements $P_n = \langle n|\rho|n\rangle$ of ρ by

$$P_m(T) = \sum_n P_n \frac{[\mu(T)]^m}{m!} \left\langle n \left| \sum_{l=0}^{\infty} \frac{\mu(T)^l}{l!} a^{\dagger m+l} a^{m+l} \right| n \right\rangle \,. \tag{3.110}$$

This gives

$$P_m(T) = \sum_{n=m}^{\infty} P_n \sum_{l=0}^{n-m} (-1)^l \frac{\mu(T)^l}{l!} \frac{n!}{(n-m-l)!} \,. \tag{3.111}$$

The l summation is equivalent to a binomial expansion and we may write [15]

$$P_m(T) = \sum_{n=m}^{\infty} P_n \binom{n}{m} [\mu(T)]^m [1-\mu(T)]^{n-m} \tag{3.112}$$

where

$$\binom{n}{m} = \frac{n!}{m!(n-m)!} .$$

This distribution is known as the *Bernoulli distribution*.

The photo-count distribution $P_n(T)$ is only the same as P_n in the case of unit quantum efficiency

$$P_m(T) = P_m, \qquad \mu(T) = 1 . \tag{3.113}$$

In practice, quantum efficiencies are less than unity and the photon-count distribution is only indirectly related to P_n.

The following results may be proved:

3.10.1 Coherent Light

$$P_n = \frac{\bar{n}^n}{n!} \exp(-\bar{n}) , \tag{3.114}$$

$$P_m(T) = \frac{[\mu(T)\bar{n}]^m}{m!} \exp[-\mu(T)\bar{n}] . \tag{3.115}$$

3.10.2 Chaotic Light

$$P_n = \frac{(\bar{n})^n}{(1+\bar{n})^{1+n}} , \tag{3.116}$$

$$P_m(T) = \frac{[\mu(T)\bar{n}]^m}{[1+\mu(T)\bar{n}]^{1+m}} . \tag{3.117}$$

These results agree with those obtained by semiclassical methods, see (3.94 and 3.103). In these cases P_n and $P_m(T)$ have the same mathematical form with the mean number $\bar{m}$ of counted photons related to the mean number $\bar{n}$ of photons in the mode by $\bar{m} = \mu(T)\bar{n}$. No such simple relation holds in general.

For example, for a photon number state, P_n is a delta function δ_{nn_0} but the photo-count distribution $P_m(T)$ is non zero for all $m \leq n_0$. However the normalized second order factorial moments are the same in all cases.

For a single-mode field

$$\sum m(m-1)\frac{P_m(T)}{\bar{m}^2} = \sum_n (n-1)\frac{P_n}{\bar{n}^2} = g^{(2)}(0) . \tag{3.118}$$

Thus the second-order correlation function $g^{(2)}(0)$ is directly obtainable from the photo-count distribution without any dependence on the quantum efficiency $\mu(T)$. For a multimode field a more complicated relation holds.

3.10.3 Photo-Electron Current Fluctuations

We now consider how the photon number statistics determines the statistics of the observed photo-electron current. Each individual photon detection produces a small current pulse, the observed current over a counting interval from $t-T$ to t is then due to the accumulated electrical pulses over this interval. Thus we write

$$i(t) = \int_{t-T}^{t} F(t')\mathrm{d}n(t') \,. \tag{3.119}$$

Here $F(t')$ is a response function which determines the current resulting from each photon detection event. We assume $F(t')$ is *flat*, i.e. independent of t,

$$F(t') = \frac{Ge}{T} \,, \tag{3.120}$$

where e is the electronic charge and G is a gain factor. Then the photo-electron current is given by

$$i(t) = \frac{Ge}{T}\, n \,, \tag{3.121}$$

where n is the total number of photon detection events over the counting interval. The mean current is then given by

$$\overline{i(t)} = \frac{Ge}{T}\sum_{n=0}^{\infty} nP_n(T;\, t) \,, \tag{3.122}$$

where $P_n(T,t)$ is given by (3.89) with

$$\bar{I}(t,\, T) = \frac{1}{T}\int_{t-T}^{t} \mathrm{d}t'\, E^{(-)}(t')E^{(+)}(t') \,. \tag{3.123}$$

Thus

$$\overline{i(t)} = (\alpha Ge)\langle : \bar{I}(t,\, T) : \rangle \,. \tag{3.124}$$

The current power spectrum is directly related to the statistical properties of the current by

$$S(\omega) = \frac{1}{\pi}\int_{0}^{\infty} \mathrm{d}\tau\, \cos\,(\omega t)\, \overline{i(0)i(\tau)} \,. \tag{3.125}$$

The two-time correlation function is determined by joint emission probabilities for photo-electrons which are generalisations of the single photon result in (3.84). Explicit expressions were given by *Carmichael* [16]. The result is, with the definitions of (3.120).

$$\overline{i(0)i(\tau)} = (\alpha Ge)^2[\langle : \bar{I}(T,\,0)\bar{I}(T,\,\tau) : \rangle + \theta(T-\tau)\langle : \bar{I}(\tau - T,\,0) : \rangle] \tag{3.126}$$

where $\theta(x)$ is zero for $x \leq 0$ and unity otherwise. For multiple time correlations :: also signifies time ordering (time arguments increasing to the left in products of annihilation operators). In the case of constant intensity

$$\begin{aligned}\overline{i(0)i(\tau)} =& (\alpha\zeta\, Ge)^2[\langle a^\dagger(0)a^\dagger(\tau)a(\tau)a(0)\rangle] \\ &+ (Ge)^2\alpha\zeta\left[\theta(T-\tau)\frac{(T-\tau)}{T^2}\langle a^\dagger(0)a(0)\rangle\right]\end{aligned} \tag{3.127}$$

where ζ is a scale factor that converts the intensity operator into a photon-flux operator. For plane waves it is given by

$$\zeta = \frac{\varepsilon_0\, cA}{\hbar\omega c} \tag{3.128}$$

where A is the transverse area of the field over which the field is measured, and ω_c the frequency of the field. Using the following result for the delta function

$$\int\limits_0^\infty \mathrm{d}t' f(t')\delta(t') = \frac{1}{2}f(0)\,, \tag{3.129}$$

one may show that

$$\lim_{T\to 0}\frac{\theta(T-t)}{T^2}(T-t) = \delta(t)\,. \tag{3.130}$$

Then in the limit of broad-band detector response $(T \to 0)$

$$\overline{i(0)i(\tau)} = (\alpha Ge\zeta)^2\langle a^\dagger(0)a^\dagger(\tau)a(\tau)a(0)\rangle + \alpha\zeta(Ge)^2\langle a^\dagger(0)a(0)\rangle\delta(\tau) \tag{3.131}$$

The last term in this expression is the *shot noise* contribution to the current.

It is more convenient to write this expression directly in terms of the normally-ordered correlation function

$$\langle : I(0), I(\tau) : \rangle \equiv \zeta^2[\langle a^\dagger(0)a^\dagger(\tau)a(\tau)a(0)\rangle - \langle a^\dagger(0)a(0)\rangle^2]\,. \tag{3.132}$$

Then

$$\begin{aligned}\overline{i(0)i(\tau)} =& (\alpha Ge\zeta)^2\langle a^\dagger(0)a(0)\rangle^2 + \alpha\zeta(Ge)^2\langle a^\dagger(0)a(0)\rangle\delta(\tau) \\ &+ (\alpha Ge)^2\langle : I(0),\, I(\tau) : \rangle\,.\end{aligned} \tag{3.133}$$

The first term is a DC term and does not contribute to the spectrum. The second term is the shot-noise contribution. The final term represents intensity fluctuations, which for a coherent field is zero.

Exercises

3.1 Calculate the mean intensity at the screen when the two slits of a Young's interference experiment are illuminated by the two photon state $(b^\dagger)^2|0\rangle/\sqrt{2}$ where $b=(a_1+a_2)/\sqrt{2}$ and a_i is the annihilation operator for the mode radiated by slit i.

3.2 In balanced homodyne detection the measured photocurrent is determined by the moments of the photon number difference at the two output ports of the beam splitter. Show that the variance of the photon-number difference for a 50/50 beam splitter is

$$V(n_-)=|\beta|^2V(X_{\theta+\pi/2})$$

where $|\beta|^2$ is the intensity of the local oscillator. Thus the local oscillator intensity fluctuations do not contribute.

3.3 Show that the probability to detect m photons with unit quantum efficiency in a field which has been transmitted by a beam splitter of transmitivity μ, is given by

$$P_m(\mu)=\sum_{n=m}^{\infty}P_n\binom{n}{m}\mu^m(1-\mu)^{n-m}$$

where P_n is the photon number distribution for the field before passing through the beam splitter.

3.4 A beam splitter transforms incoming mode operators $a_i,\ b_i$ to the outgoing operators $a_0,\ b_0$ where

$$a_0=\sqrt{\eta}a_i-i\sqrt{1-\eta}b_i,\qquad b_0=\sqrt{\eta}b_i-\mathrm{i}\sqrt{1-\eta}a_i\,.$$

(a) Show that such a transformation may be generated by the unitary operator

$$T=\exp[-\mathrm{i}\theta(a^\dagger b+ab^\dagger)],\qquad \eta=\cos^2\Theta\,.$$

(b) Thus show that if the incoming state is a coherent state $|\alpha_i\rangle\otimes|\beta_i\rangle$, the outgoing state is also a coherent state with

$$\alpha_0=\sqrt{\eta}\alpha_i-\mathrm{i}\sqrt{1-\eta}\beta_i,\qquad \beta_0=\sqrt{\eta}\beta_i-\mathrm{i}\sqrt{1-\eta}\alpha_i\,.$$

(c) Show that, if the incoming state is the product number state $|1\rangle\otimes|1\rangle$, the outgoing state is

$$(2\eta - 1)|1\rangle|1\rangle + i\sqrt{2\eta(1-\eta)}(|2\rangle|0\rangle + |0\rangle|2\rangle).$$

Note that when $\eta = 1/2$ the 'coincidence' term $|1\rangle|1\rangle$ does not appear, a result known as Hong-Ou-Mandel interference.

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Further Reading

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